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A NOTE ON THE EIGENVALUES ESTIMATES OF HERMITIAN MATRICES

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Abstract

We discuss some very simple estimates for the eigenvalues of Hermitian matrices. This also provides some information for the eigenvalues of normal matrices.

Key words: Positive linear functional, principal submatrices, Cauchy's interlacing inequalities, positive semidefinite matrices, eigenvalues, Hermitian matrices.

AMS classification: 15A42, 15B57.

1 Introduction

Let $M(n)$ denote the algebra of all complex $n \times n$ matrices. An element $A \in M(n)$ is Hermitian if $A^* = A$ where A^* denotes the conjugate transpose of A. The eigenvalues of a Hermitian matrix $A \in M(n)$ are all real and we assume that they are arranged as

$$\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A). \tag{1.1}$$

The diagonal entries of a Hermitian matrix are also real and we assume that they are enumerated as

$$a_1 \le a_2 \le \cdots \le a_n. \tag{1.2}$$

A basic result in linear algebra says that the smallest (largest) eigenvalue of a Hermitian matrix A is less (greater) than or equal to the smallest (largest) diagonal entry of A, that is

$$\lambda_1(A) \leq a_1 \text{ and } \lambda_n(A) \geq a_n. \tag{1.3}$$

A matrix $A = (a_{ij}) \in M(n)$ is nonnegative if $a_{ij} \geq 0$ for all i and j. The second smallest eigenvalue of a nonnegative symmetric matrix is less than or equal to its third smallest diagonal entry, that is

$$\lambda_2(A) \leq a_3. \tag{1.4}$$

Also, if the off-diagonal entries of a Hermitian matrix are all purely imaginary then for $k = 1, 2, ..., \lceil \frac{n-1}{2} \rceil$, we have

$$\lambda_k(A) \leq a_{2k-1} \text{ and } \lambda_{n-k+1}(A) \geq a_{n-2k+2}. \tag{1.5}$$

In this note we discuss some further estimates for the extreme eigenvalues of a Hermitian matrix. Our discussion in this note rely on the following basic principles of matrix analysis. For more details see Bhatia (1997) and Horn and Johnson (2013).

1. The Cauchy interlacing inequalities says that if A_k is any $k \times k$ principal submatrix of the Hermitian element $A \in M(n)$, then
$$\lambda_i(A) \leq \lambda_i(A_k) \leq \lambda_{i+n+k}(A) \tag{1.6}$$
for all $i = 1, 2, ..., k$.
2. The Schur majorizations inequalities says that if the eigenvalues and diagonal entries of a Hermitian matrix $A \in M(n)$ are arranged as in (1.1) and (1.2), respectively, then for $k = 1, 2, ..., n$
$$\sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} a_i \tag{1.7}$$
for $k = 1, 2, ..., n-1$ and equality holds when $k = n$.
3. A linear functional $\varphi : M(n) \to \mathbb{C}$ is said to be positive if $\varphi(A) \geq 0$ whenever A is positive semidefinite. For any Hermitian element $A \in M(n)$ we have
$$\lambda_1(A) \leq \varphi(A) \leq \lambda_n(A). \tag{1.8}$$

4. Weyl's inequalities say that if the eigenvalues of the Hermitian matrices $A, B \in M(n)$ are arranged as in (1.1), then

$$\lambda_j(A) + \lambda_1(B) \le \lambda_j(A+B) \le \lambda_j(A) + \lambda_n(B) \tag{1.9}$$

for all $j = 1, 2, ..., n$.

5. Let $\varphi_i : M(n) \to \mathbb{C}$ be positive unital linear functionals $i = 1, 2$ and A be any Hermitian element of $M(n)$, then

$$\lambda_n - \lambda_1 \ge |\varphi_1(A) - \varphi_2(A)|. \tag{1.10}$$

See Bhatia and Sharma (2014).

2 Main Results

We consider following positive unital linear functionals, See Bhatia and Sharma (2014, 2016).

$$\varphi_3(A) = a_{jj} \text{ for any fixed } j. \tag{2.1}$$

$$\varphi_4(A) = \frac{1}{n}\sum_{i,j=1}^{n} a_{ij} = \frac{\mathrm{tr}A}{n} + \frac{2}{n}\sum_{i<j}\mathrm{Re}a_{ij}. \tag{2.2}$$

$$\varphi_5(A) = \frac{\mathrm{tr}A}{n} + \frac{1}{n(n-1)}\sum_{i\neq j} a_{ij}. \tag{2.3}$$

$$\varphi_6(A) = \frac{a_{ii}+a_{jj}}{2} + i\frac{\overline{\alpha}a_{ji} - \alpha a_{ij}}{4},\ i \neq j,\ |\alpha| \le 2,\ \alpha \in \mathbb{C}. \tag{2.4}$$

$$\varphi_7(A) = \frac{a_{ii}+a_{jj}}{2} + \beta\frac{a_{ji} - a_{ij}}{4},\ i \neq j,\ \beta \in \mathbb{R},\ |\beta| \le 2. \tag{2.5}$$

Proposition 2.1. *Let $A = (a_{ij}) \in M(n)$ be Hermitian and let its eigenvalues and diagonal entries be arranged as in (1.1) and (1.2), respectively. Then*

$$\lambda_n(A) \ge a_1 + (n-1)\min_{\substack{i,j\\ i\neq j}} Re a_{ij} \ge n \min_{i,j} Re a_{ij} \tag{2.6}$$

and

$$\lambda_1(A) \le a_n + (n-1)\max_{\substack{i,j\\ i\neq j}} Re a_{ij} \le n \max_{i,j} Re a_{ij}. \tag{2.7}$$

Proof. Let $\varphi_4 : M(n) \to \mathbb{C}$ be defined as in (2.2). It is a positive unital linear functional. On applying the second inequality (1.8) to φ_4, we get

$$\lambda_n(A) \geq \frac{1}{n}\sum_{i=1}^{n} a_{ii} + \frac{1}{n}\sum_{i\neq j} a_{ij}$$
$$\geq \min_i a_{ii} + \frac{n-1}{n(n-1)}\sum_{i\neq j} a_{ij}. \tag{2.8}$$

For a Hermitian matrix $A = (a_{ij}) \in M(n)$, the arithmetic mean of $\frac{n(n-1)}{2}$ real numbers $\mathrm{Re}a_{ij}$ $(i<j)$ can be written as

$$\frac{2}{n(n-1)}\sum_{i<j}\mathrm{Re}a_{ij} = \frac{1}{n(n-1)}\sum_{i\neq j} a_{ij}.$$

The arithmetic mean of numbers x_j's lies between minimum and maximum of x_j's. Therefore

$$\min_{i,j}\mathrm{Re}a_{ij} \leq \frac{1}{n(n-1)}\sum_{i\neq j} a_{ij} \leq \max_{i,j}\mathrm{Re}a_{ij}. \tag{2.9}$$

On combining (2.8) and the first inequality (2.9) we immediately get the first inequality(2.6). The second inequality (2.6) is immediate.

Likewise, on applying the first inequality (1.8) to φ_4, we have

$$\lambda_1(A) \leq \max_i a_{ii} + (n-1)\left(\frac{1}{n(n-1)}\sum_{i\neq j} a_{ij}\right). \tag{2.10}$$

On combining the second inequality (2.9) with (2.10) we immediately get (2.7). ■

Proposition 2.2. *Let $A \in M(n)$ be Hermitian and let its eigenvalues and diagonal entries be arranged as in (1.1) and (1.2), respectively. Then*

$$\frac{1}{n-1}\sum_{i=2}^{n}\lambda_i(A) \geq a_1 - \max_{\substack{i,j\\ i\neq j}} Re a_{ij} \tag{2.11}$$

and

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\lambda_i(A) \leq a_n - \min_{\substack{i,j\\ i\neq j}} Re a_{ij}. \tag{2.12}$$

Proof. As in the proof of the Proposition 2.1, we have from the first inequality (1.8),

$$\lambda_1(A) \leq \frac{\mathrm{tr}A}{n} + \frac{1}{n}\sum_{i\neq j} a_{ij}. \tag{2.13}$$

We have $\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i(A)$. Therefore

$$\lambda_1(A) = \mathrm{tr}A - \sum_{i=2}^{n} \lambda_i(A). \tag{2.14}$$

Combining (2.13) and (2.14), we get that

$$\frac{1}{n-1}\sum_{i=2}^{n} \lambda_i(A) \geq \frac{1}{n}\mathrm{tr}A - \frac{1}{n(n-1)}\sum_{i\neq j} a_{ij}. \tag{2.15}$$

We have

$$\frac{\mathrm{tr}A}{n} \geq a_1 \text{ and } \frac{1}{n(n-1)}\sum_{i\neq j} a_{ij} \leq \max_{\substack{i,j \\ i\neq j}} \mathrm{Re}\, a_{ij}. \tag{2.16}$$

Combining (2.15) and (2.16) we immediately get (2.11). The inequality (2.12) follows on using similar arguments. ∎

Proposition 2.3. *Let $A = (a_{ij}) \in M(n)$ be nonnegative and symmetric. Then*

$$\lambda_1(A) \leq \min_{r\neq s} \left|\sqrt{a_{rr}a_{ss}} - a_{rs}\right|. \tag{2.17}$$

Proof. Let $A_2 = \begin{bmatrix} a_{rr} & a_{rs} \\ a_{rs} & a_{ss} \end{bmatrix}$ be any principal submatrix of A. We write, $A = H_1 + H_2$ where

$$H_1 = \begin{bmatrix} a_{rr} & \sqrt{a_{rr}a_{ss}} \\ \sqrt{a_{rr}a_{ss}} & a_{ss} \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} 0 & a_{rs} - \sqrt{a_{rr}a_{ss}} \\ a_{rs} - \sqrt{a_{rr}a_{ss}} & 0 \end{bmatrix}.$$

The matrix H_1 is positive semidefinite. Therefore, $\lambda_1(H_1) = 0$ and $\lambda_2(H_2) = \left|\sqrt{a_{rr}a_{ss}} - a_{rs}\right|$. Then, from the Weyl's inequalities (1.9), we have

$$\lambda_1(A_2) \leq \left|\sqrt{a_{rr}a_{ss}} - a_{rs}\right|. \tag{2.18}$$

The inequality (2.17) now follows from (2.18) on using the interlacing inequality, $\lambda_1(A) \leq \lambda_1(A_2)$. ∎

Proposition 2.4. *Let* $A=(a_{ij})\in M(n)$ *be Hermitian. Then*

$$\lambda_1(A)\leq \max_{r,s}\left\{\frac{a_{rr}+a_{ss}}{2}-|a_{rs}|\right\} \tag{2.19}$$

and

$$\lambda_n(A)\geq \max_{r,s}\left\{\frac{a_{rr}+a_{ss}}{2}+|a_{rs}|\right\}. \tag{2.20}$$

Proof. From (1.10), we find that

$$\lambda_1(A)\leq \frac{\mathrm{tr}A}{n}-\frac{1}{n}|\varphi_1(A)-\varphi_2(A)| \tag{2.21}$$

and

$$\lambda_n(A)\geq \frac{\mathrm{tr}A}{n}+\frac{1}{n}|\varphi_1(A)-\varphi_2(A)|. \tag{2.22}$$

Let $A_2=\begin{bmatrix} a_{rr} & a_{rs}\\ \overline{a_{rs}} & a_{ss}\end{bmatrix}$ be any principal submatrix of A. Then, on using (2.21) and (2.22), with $\varphi_1=\varphi_6$ and $\varphi_2=\varphi_7$ where φ_6 and φ_7 are defined in (2.4) and (2.5), respectively, we get

$$\lambda_1(A_2)\leq \frac{a_{rr}+a_{ss}}{2}-|a_{rs}| \text{ and } \lambda_2(A_2)\geq \frac{a_{rr}+a_{ss}}{2}+|a_{rs}|. \tag{2.23}$$

By interlacing inequalities (1.6), we have $\lambda_1(A)\leq\lambda_1(A_2)$ and $\lambda_n(A)\geq\lambda_2(A_2)$. The assertions of the theorem then follow on using (2.23). ■

It is clear from the above Proposition 2.4 that if $A=(a_{ij})\in M(n)$ is Hermitian and $a_{rr}\leq a_{ss}$, then for $r\neq s$,

$$\lambda_1(A)\leq \frac{a_{rr}+a_{ss}}{2}-|a_{rs}|\leq a_{ss}-|a_{rs}| \text{ and } \lambda_n(A)\geq \frac{a_{rr}+a_{ss}}{2}+|a_{rs}|\geq a_{rr}+|a_{rs}|. \tag{2.24}$$

The proof of the inequalities (2.24) also follow on using Weyl's inequalities (1.9), we have

$$\lambda_1(A)\leq\lambda_1(A_2)\leq\lambda_2\begin{bmatrix} a_{rr} & 0\\ 0 & a_{ss}\end{bmatrix}+\lambda_1\begin{bmatrix} 0 & a_{rs}\\ \overline{a_{rs}} & 0\end{bmatrix}=a_{ss}-|a_{rs}|$$

and

$$\lambda_n(A)\geq\lambda_2(A_2)\geq\lambda_1\begin{bmatrix} a_{rr} & 0\\ 0 & a_{ss}\end{bmatrix}+\lambda_2\begin{bmatrix} 0 & a_{rs}\\ \overline{a_{rs}} & 0\end{bmatrix}=a_{rr}+|a_{rs}|.$$

Proposition 2.5. *Let $A \in M(n)$ be Hermitian and let its eigenvalues and diagonal entries be arranged as in (1.1) and (1.2) respectively. Then*

$$\frac{1}{n-1}\sum_{i=2}^{n}\lambda_i(A) \geq \min_i a_{ii} + \frac{1}{n-1}\min_{i\neq j}|a_{ij}| \tag{2.25}$$

and

$$\frac{1}{n-1}\sum_{i=2}^{n}\lambda_i(A) \leq \max_i a_{ii} - \frac{1}{n-1}\max_{i\neq j}|a_{ij}|. \tag{2.26}$$

Proof. On using $\lambda_1(A) = \mathrm{tr}A - \sum_{i=2}^{n}\lambda_i(A)$ in the first inequality (2.24), we get

$$\frac{1}{n-1}\sum_{i=2}^{n}\lambda_i(A) \geq \frac{\mathrm{tr}A - a_{ss}}{n-1} + \frac{1}{n-1}|a_{rs}| \geq \min_i a_{ii} + \frac{1}{n-1}|a_{rs}|. \tag{2.27}$$

The inequality (2.27) implies (2.25).
Similarly, on using $\lambda_n(A) = \mathrm{tr}A - \sum_{i=1}^{n-1}\lambda_i(A)$ in the second inequality (2.24), we have

$$\frac{1}{n-1}\sum_{i=2}^{n}\lambda_i(A) \leq \frac{\mathrm{tr}A - a_{ss}}{n-1} - \frac{1}{n-1}|a_{rs}| \leq \max_i a_{ii} - \frac{1}{n-1}a_{rs}. \tag{2.28}$$

The inequality (2.28) implies (2.26). ∎

We show in the following theorem that something more can be said when A is positive definite matrix.

Proposition 2.6. *Let $A = (a_{ij}) \in M(n)$ be positive definite matrix. Then, for $a_{rr} \leq a_{ss}$,*

$$\lambda_1(A) \leq \min_{r,s}\left(a_{rs} - \frac{|a_{rs}|^2}{a_{ss}}\right). \tag{2.29}$$

Proof. Suppose $a_{rr} \leq a_{ss}$ and let

$$A_2 = \begin{bmatrix} a_{rr} & a_{rs} \\ \overline{a_{rs}} & a_{ss} \end{bmatrix},\ A_2^{-1} = \frac{1}{a_{rr}a_{ss} - |a_{rs}|^2}\begin{bmatrix} a_{ss} & -a_{rs} \\ -\overline{a_{rs}} & a_{rr} \end{bmatrix}.$$

On applying the second inequality (1.3) to A_2, we get

$$\lambda_2\left(A^{-1}\right) \geq \frac{a_{ss}}{a_{rr}a_{ss} - |a_{rs}|^2}. \tag{2.30}$$

We have $\lambda_2\left(A^{-1}\right) = \frac{1}{\lambda_1(A)}$. Therefore, from (2.30) on using interlacing theorem $\lambda_1(A) \leq \lambda_1(A_2)$, we immediately get (2.29). ∎

Let $A = (a_{ij}) \in M(n)$ and let λ_i's be its eigenvalues, $i = 1, 2, ..., n$. Then Hirsch (1902) proved that

$$|\lambda_k(A)| \leq n \max_{i,j} |a_{ij}|. \tag{2.31}$$

Also, see Marcus and Mink (1964).

The proof of (2.31) follows from the fact that for unit vector $x \in \mathbb{C}^n$.

$$|x^* Ax| = \left|\sum_{i,j} a_{ij}\overline{x_j}x_i\right| \leq \max_{i,j} |a_{ij}| \sum_{i,j} |x_i| |x_j| = \max_{i,j} |a_{ij}| \left(\sum_{i=1}^{n} |x_i|\right)^2$$

and by the Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} |x_i|\right)^2 \leq n \sum_{i=1}^{n} |x_i|^2 = n.$$

So, $|x^* Ax| \leq n \max_{i,j} |a_{ij}|$ and since $|\lambda_k(A)| \leq |x^* Ax|$, we immediately get (2.31).

It is well known that $\sum_{i=1}^{n} \lambda_i(A) = \sum_{i=1}^{n} a_{ii} = \mathrm{tr}A$ where $\mathrm{tr}A$ denotes the trace of A. We then have $\lambda_1(A) \leq \frac{\mathrm{tr}A}{n} \leq \lambda_n(A)$. If all the eigenvalues are nonnegative, we have

$$\lambda_k(A) \leq \frac{\mathrm{tr}A}{n-k+1} \tag{2.32}$$

for all $k = 1, 2, .., n$.

We prove a related result in the following proposition.

Proposition 2.7. *Let $A \in M(n)$ be Hermitian. Let c_k be the largest absolute entry of any $k \times k$ principal of A. Then*

$$\lambda_k(A) \leq kc_k \tag{2.33}$$

for all $k = 1, 2, ..., n$.

Proof. By the interlacing inequalities (1.6) and the inequality (2.31), we have $\lambda_k(A) \leq \lambda_k(A_k) \leq kc_k$. ∎

Let $A \in M(n)$ be positive semidefinite. Then the largest diagonal entry of A is greater than or equal to its largest absolute off-diagonal entry. It follows that the kth smallest diagonal entry a_k of A is the largest absolute entry of some $k \times k$ principal submatrix of A. Thus, if $A = (a_{ij}) \in M(n)$ is positive semidefinite, then from (2.33),

$$\lambda_k(A) \leq ka_k$$

for all $k = 1, 2, ..., n$.

On using the Schur majorization inequalities (1.7) we have the following refinement of (2.32) and (2.33) for positive semidefinite matrices,

$$\lambda_k (A) \leq \sum_{i=1}^{k} \lambda_i (A) \leq \sum_{i=1}^{k} a_i \leq k a_k. \tag{2.34}$$

We can use the above inequalities to derive some results for the real and imaginary parts of the eigenvalues of normal matrices. Let

$$B = \frac{A + A^*}{2} \text{ and } C = \frac{A - A^*}{2i}.$$

Then, if A is normal

$$\text{Re}\lambda (A) = \lambda \left(\frac{A + A^*}{2} \right) \text{ and } \text{Im}\lambda (A) = \lambda \left(\frac{A - A^*}{2i} \right).$$

So, if $A = (a_{ij}) \in M (n)$ is normal. Then, from (2.6) and (2.7), we have

$$\max_i \text{Re}\lambda_i (A) \geq n \min_{i,j} \text{Re} (a_{ij} + a_{ji})$$

and

$$\min_i \text{Re}\lambda_i (A) \leq n \max_{i,j} \text{Re} (a_{ij} + a_{ji}) .$$

Likewise, we can obtain the related inequalities for real and imaginary parts of the eigenvalues of a normal matrix using the above inequalities derived for Hermitian matrices.

3 Examples

We first give an example to illustrate that on using Cauchy's interlacing inequalities we can estimate the eigenvalues of the matrix on using estimates for the eigenvalues of its principal submatrix. This example shows that sometimes such simple estimates may give surprisingly better estimates.

Example 1. Let $A = (a_{ij}) \in M (n)$ be a symmetric matrix and let $a_{ij} = 2j - i$,

$j = 1, 2, ..., i$. Then,

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\ 0 & 2 & 1 & 0 & -1 & -2 & -3 & -4 & -5 & -6 \\ -1 & 1 & 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \\ -2 & 0 & 2 & 4 & 3 & 2 & 1 & 0 & -1 & -2 \\ -3 & -1 & 1 & 3 & 5 & 4 & 3 & 2 & 1 & 0 \\ -4 & -2 & 0 & 2 & 4 & 6 & 5 & 4 & 3 & 2 \\ -5 & -3 & -1 & 1 & 3 & 5 & 7 & 6 & 5 & 4 \\ -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 7 & 6 \\ -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 & 9 & 8 \\ -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 & 10 \end{bmatrix}_{10\times 10}.$$

Walker and Mieghan (2008) have used costly estimates and have shown that $\lambda_{10}(A) \geq 22.16$, while the Weyl's inequality (1.9) we have $\lambda_{10}(A) \geq 5.5$. But if we apply (1.9) to the principal submatrix

$$A_5 = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 \\ 5 & 7 & 6 & 5 & 4 \\ 4 & 6 & 8 & 7 & 6 \\ 3 & 5 & 7 & 9 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix},$$

we get a better estimate $\lambda_{10}(A) \geq \lambda_5(A_5) \geq 28$.

Example 2. Further, we consider one more example of Wolkowicz and Styan (1980). Let

$$C = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix}.$$

Wolkowicz and Styan (1980) have used bounds for eigenvalues using traces and have shown that $\lambda_5(C) \geq 7.449$. But on using the second inequality (1.3) we immediately have a better estimates $\lambda_5(C) \geq 8$.

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